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AN INTEGRAL OF PRODUCTS OF LEGENDRE FUNCTIONS AND A CLEBSCH-GOR--ETC(U)

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UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

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AND A CLEBSCH-GORDAN SUM

Richard Askey*

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ABSTRACT

New proofs and extensions are given of a sum considered by A. M. Din involving Clebsch-Gordan coefficients with zero magnetic quantum numbers and of an integral involving the product of three Legendre functions, one of the second kind.

AMS (MOS) Subject Classifications: 33A45, 33A70, 81.33

Key Words: Legendre polynomials, Legendre function, Clebsch-Gordan coefficients

Work Unit Number 1 - Applied Analysis

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AN INTEGRAL OF PRODUCTS OF LEGENDRE FUNCTIONS
AND A CLEBSCH-GORDAN SUM

Richard Askey*

Din [1] showed that

$$S := \sum_{\substack{i=|c-b| \\ i \neq a}}^{c+b} \frac{2i+1}{i(i+1) - a(a+1)} (c_{i0b0}^{c0})^2 = 0 \quad (1)$$

when a, b and c are non-negative integers with $a + b + c$ odd and $|c-b| \leq a \leq c+b$. The Clebsch-Gordan coefficients with zero magnetic quantum numbers are given by

$$(c_{i0b0}^{c0})^2 = \frac{2c+1}{2} \int_{-1}^1 dx P_i(x) P_b(x) P_c(x), \quad (2)$$

This integral was evaluated by Ferrers and others in the last century. The evaluation comes from the linearization formula

$$P_n(x) P_m(x) = \sum_{k=0}^{\min(m,n)} \frac{(\frac{1}{2})_{m-k} (\frac{1}{2})_{n-k} (\frac{1}{2})_k (m+n-k)! (m+n-2k+\frac{1}{2})}{(m-k)! (n-k)! k! (\frac{1}{2})_{m-n-k} (m+n-k+\frac{1}{2})} P_{m+n-2k}(x), \quad (3)$$

and the orthogonality of Legendre polynomials. See [2]. To show (1) Din reduced it to showing that

$$I(a,b,c) := \int_{-1}^1 dx Q_a(x) P_b(x) P_c(x) = 0 \quad (4)$$

when $a, b, c \geq 1$ are integers, $a + b + c$ is odd and $|c-a| < b < c+a$. Here $P_i(x)$ is the Legendre polynomial and $Q_a(x)$ is the Legendre function of the second kind on the cut $[-1,1]$. He ended the paper by stating that I could evaluate (4) for general integers a, b, c . The details follow.

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Din started with

$$\int_{-1}^1 dx Q_a(x) P_b(x) = \frac{1 - \cos(b-a)\pi}{(b-a)(b+a+1)}, \quad a, b = 1, 2, \dots, a \neq b, \quad (5)$$

with a reference to [3]. A generalization of (5) is given there when a and b are complex, $\operatorname{Re} a > 0$, $\operatorname{Re} b > 0$, and the extra term which occurs vanishes when either a or b is an integer. The argument in [3] used the Legendre differential equation. Here is a second derivation of (5). Start with an expansion of Heine [4]

$$Q_a(\cos \theta) = \frac{2|a|}{(\frac{3}{2})_a} \sum_{i=0}^{\infty} \frac{(\frac{1}{2})_i (a+1)_i}{i! (a + \frac{3}{2})_i} \cos(a+2i+1)\theta.$$

The shifted factorial $(c)_n$ is defined by

$$(c)_n = \Gamma(n+c)/\Gamma(c) = c(c+1)\cdots(c+n-1).$$

Since $P_a(-x) = (-1)^a P_a(x)$ and $Q_a(-x) = (-1)^{a+1} Q_a(x)$, $a = 0, 1, \dots$, we may assume a and b have opposite parity, for the integral in (5) vanishes when a and b have the same parity. Then

$$\begin{aligned} I(a, b, 0) &= \frac{2|a|}{(\frac{3}{2})_a} \sum_{i=0}^{\infty} \frac{(\frac{1}{2})_i (a+1)_i}{i! (a + \frac{3}{2})_i} \int_0^\pi d\theta \cos(a+2i+1)\theta \sin \theta P_b(\cos \theta) \\ &= \frac{2|a|}{(\frac{3}{2})_a} \sum_{i=0}^{\infty} \frac{(\frac{1}{2})_i (a+1)_i (a+2i+1) (i+(a+b-1)/2) (-\frac{1}{2})_{i+(a+1-b)/2}}{i! (a + \frac{3}{2})_i (\frac{3}{2})_{i+(a+b+1)/2} (i+(a+1-b)/2)!} \end{aligned}$$

by a special case of an integral of Gegenbauer which is equivalent to [5]

$$C_n^\mu(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(\mu)_{n-k} (\mu-\lambda)_k (n-2k+\lambda)}{(\lambda+1)_{n-k} k! \lambda} C_{n-2k}^\lambda(x),$$

where $C_n^\lambda(x)$ is the ultraspherical polynomial.

The above sum can be written as a generalized hypergeometric series and then summed by a formula of Dougall [6]. A more general sum of Dougall will be stated below. A routine reduction shows that (5) holds when $a, b = 0, 1, \dots$, with the integral equal to zero when $a = b$.

To compute the evaluation of (4) use the Ferrers-Adams linearization formula (3) and (5) to obtain

$$I(a,b,c) = \sum_{k=0}^{\min(b,c)} \frac{(\frac{1}{2})_{b-k} (\frac{1}{2})_{c-k} (\frac{1}{2})_k (b+c-k)! (b+c-2k+\frac{1}{2})}{(b-k)! (c-k)! k! (\frac{1}{2})_{b+c-k} (b+c-k+\frac{1}{2})}$$

$$\frac{[1-\cos(b+c-2k-a)\pi]}{(b+c-a-2k)(b+c+a+1-2k)}$$

$$= \frac{[1-\cos(b+c-a)\pi] (\frac{1}{2})_b (\frac{1}{2})_c (b+c)!}{(b+c-a)(b+c+a+1)b!c! (\frac{1}{2})_{b+c}}$$

$${}_7F_6 \left(\begin{matrix} -b-c-\frac{1}{2}, -b/2-c/2+\frac{3}{4}, -b, -c, \frac{1}{2}, (a-b-c)/2, (-1-a-b-c)/2 \\ -b/2-c/2-\frac{1}{4}, \frac{1}{2}-c, \frac{1}{2}-b, -b-c, (1-a-b-c)/2, (2+a-b-c)/2 \end{matrix} ; 1 \right)$$

Dougall's sum of the very well poised 2-balanced ${}_7F_6$ [7],

$${}_7F_6 \left(\begin{matrix} a, 1+a/2, b, c, d, e, -n \\ a/2, 1+a-b, 1+a-c, 1+a-d, 1+a-e, 1+a+n \end{matrix} ; 1 \right)$$

$$= \frac{(1+a)_n (1+a-b-c)_n (1+a-b-d)_n (1+a-c-d)_n}{(1+a-b)_n (1+a-c)_n (1+a-d)_n (1+a-c-d)_n} \quad (8)$$

when $1+2a = b+c+d+e-n$, can be used and the result is

$$\int_{-1}^1 dx Q_a(x) P_b(x) P_c(x)$$

$$= \frac{[1-\cos(b+c-a)\pi] (-(b+c+a)/a)_c ((b-c-a+1)/2)_c}{(b+c-a)(b+c+a+1) (-(b+c+a-1)/2)_c ((b-c-a)/2)_c} \quad (9)$$

when $0 \leq b \leq c$, $a+b+c$ odd, and zero when $a+b+c$ is even. Since this integral vanishes when $b+c+a$ is even, we may write $a = b+c+1+2k$. The integral is then

$$\int_{-1}^1 dx Q_{b+c+1+2k}(x) P_b(x) P_c(x) = - \frac{\Gamma(k+b+c+\frac{3}{2})\Gamma(k+b+1)\Gamma(k+c+1)\Gamma(k+\frac{1}{2})}{2\Gamma(k+b+c+2)\Gamma(k+b+\frac{3}{2})\Gamma(k+c+\frac{3}{2})\Gamma(k+1)} \quad (10)$$

This integral vanishes when $k = -1, -2, \dots, -\min(b, c)$ as was shown by Din.

Since (5) holds when a is not an integer, and the rest of the above argument only used the integrality of b and c , formula (8) continues to hold when $\operatorname{Re} a \geq 0$. In this case it is better to write it as

$$\int_{-1}^1 dx Q_a(x) P_b(x) P_c(x) = \frac{[1 - \cos(b+c-a)\pi] \cdot \Gamma(\frac{c-b-a}{2})\Gamma(\frac{b-c-a}{2})}{(b+c-a)(b+c+a+1)\Gamma(\frac{c-b-a+1}{2})\Gamma(\frac{b-c-a+1}{2})} \quad (11)$$

$$\frac{\Gamma(\frac{b+c-a+1}{2})\Gamma(\frac{-b-c-a+1}{2})}{\Gamma(\frac{b+c-a}{2})\Gamma(\frac{-b-c-a}{2})}, \quad \operatorname{Re} a \geq 0, b, c = 0, 1, \dots,$$

with an appropriate limit taken when one of the gamma functions has a pole.

The sum in (1) can be evaluated in exactly the same way, only the details are easier. One only needs to use (2) to replace the Clebsch-Gordan coefficients by a known integral, rewrite the series as a generalized hypergeometric series and use Dougall's sum (8). Fortunately Din was unaware of Dougall's sum, for the integral in (11) seems to be a fundamental result, and it does not seem to have been evaluated before. I was surprised by this, since Hobson [8] wrote that F. E. Neumann had evaluated this integral. However it is not given in the book of Neumann that Hobson mentions nor in the other book of Neumann that I have looked at.

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